

# Heat, temperature and relativity

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June 15, 2009

*Dedicated to Professor Leszek Wojtczak*

The present work is motivated by still ongoing controversies on such fundamental concepts of relativistic thermodynamics as heat, work, temperature or probability (see e.g. [1] and references therein, and [2]). We are going to show how a distinguished paper by Cubero et al [3] sheds new light on these controversies and enables us to solve them to some extent. In [3] the authors present their results on one-dimensional relativistic particle dynamics simulations. They have shown that numerical results of the simulations are in excellent agreement with the *Jüttner distribution* [4] if the thermodynamic system (ideal gas) is at rest with respect to the inertial laboratory frame or with some counterpart of Jüttner distribution if this gas moves with a constant velocity. Assuming that the same is true in three dimensions one finds that the relativistic generalization of the Maxwell distribution for ideal gas is given by the *Jüttner distribution* [4]

$$dw_{\vec{p}}^{(J)} = \frac{1}{Z} \exp\{-\beta c \sqrt{\vec{p}^2 + m^2 c^2}\} dp_x dp_y dp_z \quad (1)$$

if the velocity of the gas  $\vec{V} = 0$ , and for any  $\vec{V}$  by

$$dw_{\vec{p}} = \frac{1}{Z \gamma(\vec{V})} \exp\{-\beta c u_j p^j\} dp_x dp_y dp_z \quad (2)$$

where  $Z$  is the normalization constant,  $\vec{p} = (p_x, p_y, p_z)$  is momentum of the particle;  $p^j$ ,  $j = 1, 2, 3, 4$ , stands for the four-vector of momentum i.e.,  $p^j = (\vec{p}, \frac{\mathcal{E}}{c})$ ,  $\mathcal{E} = c \sqrt{\vec{p}^2 + m^2 c^2}$ ;  $u^j$  is the four-velocity of thermodynamic system i.e.,  $u^j = (\gamma(\vec{V}) \frac{\vec{V}}{c}, \gamma(\vec{V}))$ ,  $\gamma(\vec{V}) = \frac{1}{\sqrt{1 - \frac{\vec{V}^2}{c^2}}}$ ; finally  $\beta = \frac{1}{k T_0}$ , where  $k$  is the Boltzmann constant and  $T_0$  is the gas temperature in the rest frame of the gas.

From (2) we quickly infer that the relativistic Gibbs distribution reads

$$dw = \frac{V^N}{(2\pi\hbar)^{3N} N! Z} \exp\{-\beta c u_j P^j\} d^{3N} p \quad (3)$$

where  $V$  is the volume of the system,  $N$  denotes the number of particles,  $P^j = (\vec{P}, \frac{E}{c})$  is the total four-momentum and  $\mathcal{Z}$  stands for the partition function

$$\mathcal{Z} = \frac{V^N}{(2\pi\hbar)^{3N} N!} \int \exp\{-\beta c u_j P^j\} d^{3N}p \quad (4)$$

Then the entropy  $S$  reads

$$S = -k \langle \ln(\frac{1}{\mathcal{Z}} \exp\{-\beta c u_j P^j\}) \rangle = k(\ln \mathcal{Z} + \beta c u_j \langle P^j \rangle) \quad (5)$$

and it can be shown by direct calculations (see for example [5]) that the partition function  $\mathcal{Z}$ , the entropy  $S$  and the pressure  $P$  of the gas are relativistic invariants. Moreover, straightforward calculations lead to the relations [1],[5, 6, 7, 8, 9]

$$\langle E \rangle + PV = \gamma(\vec{V})(\langle E_0 \rangle + P_0 V_0), \quad \langle \vec{P} \rangle = \gamma(\vec{V})(\langle E_0 \rangle + P_0 V_0) \frac{\vec{V}}{c^2} \quad (6)$$

which say that  $(\langle \vec{P} \rangle, \frac{\langle E \rangle + PV}{c})$  constitutes a four-vector. Consequently,  $(\langle \vec{P} \rangle, \frac{\langle E \rangle}{c})$  is not a four-vector, but according to (5)  $u_j \langle P^j \rangle$  is a Lorentz invariant. In the paper the subindex "0" stands for the physical quantities in the rest frame of the system.

From (6) one gets immedietly the first law of thermodynamics in special relativity in the form

$$d\langle P^j \rangle = \frac{1}{c} T_0 u^j dS - \frac{1}{c} \delta_4^j d(PV) + \frac{1}{c} \gamma(\vec{V}) V u^j dP + \frac{\langle E_0 \rangle + P_0 V_0}{c} du^j \quad (7)$$

Hence, we conclude from (7) that the four-vectors of temperature  $T^j$  and heat  $\delta Q^j$  are defined by

$$T^j := \frac{1}{c} T_0 u^j, \quad \delta Q^j := T^j dS \quad (8)$$

and the four-object of relativistic work  $\delta L^j$  (which is not a four-vector!) reads

$$\delta L^j = -\frac{1}{c} \delta_4^j d(PV) + \frac{1}{c} \gamma(\vec{V}) V u^j dP + \frac{\langle E_0 \rangle + P_0 V_0}{c} du^j \quad (9)$$

Consequently, the temperature  $T = cT_4$  and the heat  $\delta Q = c\delta Q_4$  transform as follows

$$T = \gamma(\vec{V}) T_0, \quad \delta Q = \gamma(\vec{V}) \delta Q_0 \quad (10)$$

as has been proved by H. Ott [10], H. Arzeliés [11] and C. Møller [8, 9] (see also [1], [12], [13]). Then the work  $\delta L = c\delta L_4$  takes the form

$$\delta L = -PdV + (\gamma(\vec{V}))^2 \frac{\vec{V}^2}{c^2} V dP + (\langle E_0 \rangle + P_0 V_0) d\gamma(\vec{V}) \quad (11)$$

(compare [1], Eq. (44)).

Now we find the first law of thermodynamics directly from the Gibbs distribution (3). Standard calculations of statistical thermodynamics (see for example [14]) give

$$u_j d\langle P^j \rangle = \frac{1}{kc\beta} dS + u_j \langle dP^j \rangle \quad (12)$$

From (12) one infers that  $d\langle P^j \rangle - \langle dP^j \rangle$  is a four-vector, and

$$d\langle P^j \rangle = \frac{1}{kc\beta} u^j dS + \langle dP^j \rangle + \delta Q_\perp^j \quad (13)$$

where  $\delta Q_\perp^j$  is a four-vector of heat orthogonal to  $u^j$  i.e.

$$u_j \delta Q_\perp^j = 0. \quad (14)$$

Assuming that in the rest frame of gas, for any reversible process one has (see (9) for  $\vec{V} = 0$  and Ref. [8])

$$\delta Q_{0\perp}^4 = 0, \quad \delta Q_{0\perp}^\mu = -\langle dP_0^\mu \rangle = 0, \quad \mu = 1, 2, 3, \quad (15)$$

we rewrite (13) in the following form

$$d\langle P^j \rangle = \frac{1}{kc\beta} u^j dS + \langle dP^j \rangle \quad (16)$$

Equation (16) is the first law of thermodynamics derived from the relativistic Gibbs distribution (3). Therefore

$$\delta Q^j = \frac{1}{kc\beta} u^j dS \quad \implies \quad T^j = \frac{1}{kc\beta} u^j, \quad T = \frac{1}{k\beta} \gamma(\vec{V}) \quad (17)$$

according to (8) and (10), and as usually

$$\delta L^j = \langle dP^j \rangle. \quad (18)$$

Relativistic temperature  $T$  given by (10) can be measured with the use of relativistic Carnot cycle [9],[1]. To see this we consider a thermodynamic engine being a slightly simplified version of the one analysed by Møller [9]. The engine realizes the relativistic Carnot cycle and it operates between two reservoirs  $R_0$  and  $R$ . The reservoir  $R$  moves with a constant velocity  $\vec{V}$  with respect to  $R_0$ . The temperature of both  $R_0$  and  $R$  in their rest frames is  $T_0$  and the temperature of  $R$  with respect to the rest frame of  $R_0$  is, by (10),  $T = \gamma(\vec{V})T_0$ . The engine works as follows

- (I) The amount of heat  $Q_0$  is absorbed isothermically from  $R_0$  at the temperature  $T_0$ .
- (II) The system is accelerated adiabatically to the velocity  $\vec{V}$ .

- (III) The amount of heat  $Q = \gamma(\vec{V})Q_0$  (with respect to the rest frame of  $R_0$ !) is released isothermically from the system to  $R$  at the temperature  $T = \gamma(\vec{V})T_0$  (with respect to the rest frame of  $R_0$ !).
- (IV) Finally, the system is decelerated adiabatically so that it returns to the initial state.

From the second law of thermodynamics it follows that

$$\frac{Q}{Q_0} = \frac{T}{T_0} = \gamma(\vec{V}) \quad (19)$$

and this *enables us to check experimentally the validity of the transformation rules* (10). Observe also that the efficiency of our Carnot cycle reads

$$\eta = \frac{T_0 - T}{T_0} = 1 - \gamma(\vec{V}) < 0. \quad (20)$$

If the transformation rules were  $T = (\gamma(\vec{V}))^{-1}T_0$ ,  $\delta Q = (\gamma(\vec{V}))^{-1}\delta Q_0$ , as was assumed by M. Planck, K.V. Mosengeil, M.V. Laue, W. Pauli or A. Einstein (who finally changed his opinion in 1952/53) (see [1]), then the efficiency would be

$$\eta = 1 - \frac{1}{\gamma(\vec{V})} > 0. \quad (21)$$

In contrary to phenomenological thermodynamics in relativistic statistical thermodynamics the transformation rule for temperature depends on the definition of "statistical thermometer" and there exists no natural, uniquely defined rule of transformation. For example if you assume that by comparing (2) with (1) the temperature should be defined as

$$T := \frac{1}{k\beta u^4} = \frac{1}{\gamma(\vec{V})}T_0 \quad (22)$$

you get the transformation rule of M. Planck, K.V. Mosengeil and others. Moreover, as can be easily shown by using the Bose-Einstein counterpart of (2) for black body radiation the temperature of a moving black body is not well defined [15],[16].

[Remark: After preparing this paper for publication I found Ref. [17] where relativistic thermodynamics from the point of view of Jüttner distribution has been also considered.]

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